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DIFFRACTION OF PROGRESSING WAVES BY EDGES

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R. M. Lewis

Work Performed for  
ADVANCED RESEARCH PROJECTS AGENCY  
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DEVELOPMENT ENGINEERING DIVISION  
DIRECTORATE OF PLANNING AND TECHNOLOGY  
ELECTRONIC SYSTEMS DIVISION  
AIR FORCE SYSTEMS COMMAND  
UNITED STATES AIR FORCE  
L. G. Hanscom Field, Bedford, Massachusetts



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Advanced Research Projects Agency  
Project Defender  
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Project 8051

Prepared by

THE MITRE CORPORATION  
Bedford, Massachusetts  
Contract AF19(628)-5165

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## FOREWORD

The work reported in this document was performed by The MITRE Corporation, Bedford, Massachusetts, for Advanced Research Projects Agency; the contract was monitored by the Directorate of Planning and Technology, Electronic Systems Division, Air Force Systems Command, under Contract AF 19(628)-5165.

## REVIEW AND APPROVAL

Publication of this technical report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

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Project Officer  
Development Engineering Division  
Directorate of Planning & Technology

## ABSTRACT

For high frequency monochromatic waves the geometrical theory of diffraction can be used for predicting radar scattering. For typical targets the strongest returns, due to speculars, occur only at special aspect angles. At most aspect angles the dominant returns are usually due to diffraction by the "edges" of the target.

The proposed use of short pulse radars requires the consideration of scattering by non-monochromatic signals. The progressing wave formalism is a generalization of the geometrical theory of diffraction suitable for treating pulses for which the high frequencies predominate. This paper extends the earlier work on specular scattering of progressing waves to include scattering by edges. For a moving target, partial results are obtained.

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## SECTION I

### INTRODUCTION

The "geometrical theory of diffraction"<sup>[1]</sup> is useful for the solution of problems of scattering of time-harmonic waves at high frequencies. A recent generalization of that theory, the "progressing wave formalism"<sup>[2]</sup> is suitable for scattering of pulses of arbitrary form by fixed or moving objects. In both theories the largest returns (specular returns) are due to reflected waves, but for typical objects specular returns occur only at special aspect angles, and at other angles the main returns are usually due to diffraction by edges.

The purpose of this paper is to derive the formulas for the diffracted progressing wave produced by an arbitrary edge. The reflected wave is discussed in [2]. The theory is based on the scalar wave equation, but the generalization to the electromagnetic case should be straight-forward (see [3] and [4]). In Section II we discuss the progressing wave formalism for the wave equation. This discussion is a simplified and improved version of the basic theory presented in [2]. The use of analytical formulas (involving jacobians) rather than geometrical formulas (involving area ratios) for the amplitude coefficients is a great convenience. An improvement in the theory is also achieved by the use of finite part integrals, which were recently exploited. In Section III we derive the formulas for the diffracted wave. The method used relies heavily

on the geometrical theory of diffraction, but there are several new features that do not appear in the time-harmonic case. A special case of the formula derived was obtained earlier in [5], but there only "time-independent" progressing waves (see Section II) were treated and the discussion was restricted to the two-dimensional case. In Section IV we obtain partial results for diffraction by a moving edge. (A complete theory would require the solution of a non-trivial "canonical problem"). These results show some interesting new feature involving the direction of the diffracted ray and a "Doppler shift" in the instantaneous frequency. The appendix contains a discussion of the fractional integration operator, which is essential to the theory of edge diffraction.



## SECTION II

### THE PROGRESSING WAVE FORMALISM

Let

$$v(t, \underline{x}) = e^{-i\omega t} u(\underline{x}) \quad (1)$$

be a solution of the wave equation

$$Lv = \nabla_{\underline{x}} \nabla_{\underline{x}} v - \frac{1}{c^2(\underline{x})} v_{tt} = 0 \quad (2)$$

Here  $\omega$  is a constant and the repeated index  $\nu$  is summed from 1 to 3.

By inserting (1) in (2) we find that  $u$  satisfies the reduced wave equation (for simplicity we take  $c$  to be constant)

$$\nabla_{\underline{x}} \nabla_{\underline{x}} u + k^2 u = 0, \quad k = \frac{\omega}{c} \quad (3)$$

Contemporary generalizations of geometrical optics are often based on asymptotic solutions of (3) of the form

$$u(\underline{x}) \sim e^{iks(\underline{x})} \sum_{m=0}^{\infty} (ik)^{-m} z_m(\underline{x}), \quad k \rightarrow \infty \quad (4)$$

If we insert (4) into (1), multiply by an arbitrary function  $a(\omega)$ , and integrate with respect to  $\omega$ , we obtain a formal solution of (2) of the form

$$v(t, \underline{x}) = \sum_{m=0}^{\infty} e_m \left[ t - \frac{s(\underline{x})}{c} \right] z^{(m)}(\underline{x}) . \quad (5)$$

Here

$$z^{(m)}(\underline{x}) = (-c)^m z_m(\underline{x}) \quad (6)$$

and

$$e_m(t) = \int a(\omega) (-i\omega)^{-m} e^{-i\omega t} d\omega . \quad (7)$$

We note that the functions  $e_m(t)$  are successive anti-derivatives, i.e.,

$$\dot{e}_m(t) = e_{m-1}(t) . \quad (8)$$

A solution of (2) of the form (5) is called a time-independent progressing wave, provided (8) is satisfied. Since (4) is valid for large  $k = \omega/c$ , it is clear that (5) is valid provided the spectrum  $a(\omega)$  contains only high frequencies, or at least  $a(\omega)$  is small except for large  $\omega$ . This provides an interpretation of the formal expansions we shall construct. Other interpretations are discussed in [2] .

We now generalize (5). A (general) progressing wave solution of (2) is given by an expansion of the form

$$v(t, \underline{x}) = \sum_{m=0}^{\infty} e_m [\phi(t, \underline{x})] z^{(m)}(t, \underline{x}) \quad (9)$$

provided (8) is satisfied. The function  $e_0(t)$  is called the wave-form and is essentially arbitrary. We assume that all of the  $e_m(t)$  vanish at  $t = -\infty$ . Then the successive  $e_m(t)$  are determined uniquely by (8).

In order to determine the phase function  $\phi(t, \underline{x})$  and the amplitude functions  $z^{(m)}(t, \underline{x})$ , we insert (9) into (2) and use (8). This yields the characteristic equation

$$\phi_{x_{\vee}} \phi_{x_{\vee}} - \frac{1}{c^2(\underline{x})} \phi_t^2 = 0 \quad (10)$$

and the transport equations

$$2 [\phi_{x_{\vee}} z_{x_{\vee}}^{(m)} - \frac{1}{c} \phi_t z_t^{(m)}] + z^{(m)} L\phi = -Lz^{(m-1)},$$

$$m = 0, 1, 2, \dots; \quad z^{(-1)} \equiv 0. \quad (11)$$

We introduce the functions

$$\omega = -\phi_t(t, \underline{x}), \quad k_{\vee} = \phi_{x_{\vee}}(t, \underline{x}) \quad (12)$$

and set

$$\underline{k} = (k_1, k_2, k_3), \quad k = \sqrt{k_v k_v} . \quad (13)$$

Then (10) takes the form

$$\omega^2 = c^2 k^2 \quad (14)$$

or

$$\omega = h(\underline{k}, \underline{x}) = \pm c(\underline{x})k . \quad (15)$$

The first order partial differential equation (15) can be solved by the method of characteristics. We introduce the characteristic equations (Hamilton's equation)

$$\dot{x}_v = g_v, \quad \dot{k}_v = - \frac{\partial h}{\partial x_v} \quad (v = 1, 2, 3) \quad (16)$$

where  $\underline{g} = (g_1, g_2, g_3)$  is the group velocity vector given by

$$g_v = \frac{\partial h}{\partial k_v} = \pm c k_v / k = c^2 k_v / \omega \quad (v = 1, 2, 3) . \quad (17)$$

A solution  $\underline{x}(t)$ ,  $\underline{k}(t)$  of the system of ordinary differential equations (16) determines a curve  $\underline{x} = \underline{x}(t)$  in  $\underline{x}$ -space called a ray. This ray is the projection onto  $\underline{x}$ -space of the space-time curve  $[t, \underline{x}(t)]$ .

The latter is called a characteristic curve of (10), or bicharacteristic of (2). We note that

$$\dot{\omega} = \partial h / \partial k_{\nu} \dot{k}_{\nu} + \partial h / \partial x_{\nu} \dot{x}_{\nu} = 0 \quad (18)$$

and

$$\dot{\phi} = \phi_t + \phi_{x_{\nu}} \dot{x}_{\nu} = -\omega + k_{\nu} g_{\nu} = \frac{1}{\omega} (c^2 k^2 - \omega^2) = 0, \quad (19)$$

i.e.  $\omega$  and  $\phi$  are constant on bicharacteristics. Thus the bicharacteristics lie on level surfaces of  $\phi(t, \underline{x})$ . These level surfaces are called characteristic hypersurfaces of (2). If the value of  $\phi$  is prescribed on some hypersurface or lower dimensional manifold  $M$ , the values of  $\omega$ , and  $\underline{k}$  can then be obtained by differentiation along  $M$ . Then the bicharacteristics emanating from  $M$  can be obtained by solving the initial value problem for (16), and from (19) the value of  $\phi$  on each bicharacteristic is equal to its value on  $M$ . This procedure for determining the value of  $\phi(t, \underline{x})$  will be illustrated in Sections III and IV. It is also discussed in detail in [2].

In order to determine the function  $z^{(m)}$  we write (11) in the form

$$\dot{z}^{(m)} + \left( \frac{c^2}{2\omega} L\phi \right) z^{(m)} = z_t^{(m)} + g_{\nu} z_{x_{\nu}}^{(m)} + \left( \frac{c^2}{2\omega} L\phi \right) z^{(m)} = - \frac{c^2}{2\omega} Lz^{(m-1)}. \quad (20)$$

Thus we see that the transport equations are in fact first order linear ordinary differential equations along the bicharacteristics. A brief digression is now required in order to obtain the solutions  $z^{(m)}$

in a convenient form. Let us assume that we have a 3-parameter family of bicharacteristics  $[t, \underline{x}(t, \underline{\sigma})]$  where  $\underline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  is the parameter vector (e.g.  $\sigma$  might be the parameter vector in a parametric representation of the hypersurface  $M$ ). We introduce the jacobian

$$j = j(t, \underline{\sigma}) = \det \left[ \frac{\partial x_i}{\partial \sigma_v} (t, \underline{\sigma}) \right] = \frac{\partial(x_1, x_2, x_3)}{\partial(\sigma_1, \sigma_2, \sigma_3)} \quad (21)$$

and observe that

$$\sum_{v=1}^3 \frac{\partial x_k}{\partial \sigma_v} \text{cof} \frac{\partial x_i}{\partial \sigma_v} = j \delta_{ik} . \quad (22)$$

Here  $\delta_{ik}$  is the Kronecker symbol and "cof" denotes the cofactor. If  $i = k$  (22) follows from the rule for the expansion of a determinant by cofactors. If  $i \neq k$ , it follows from the fact that a determinant with two identical rows vanishes. By differentiating the determinant we find that

$$\frac{dj}{dt} = \sum_{i,v} \frac{d}{dt} \left( \frac{\partial x_i}{\partial \sigma_v} \right) \text{cof} \frac{\partial x_i}{\partial \sigma_v} = \sum_{i,v,k} \left[ \frac{\partial}{\partial x_k} \frac{dx_i}{dt} \right] \frac{\partial x_k}{\partial \sigma_v} \text{cof} \frac{\partial x_i}{\partial \sigma_v} = j \sum_i \frac{\partial g_i}{\partial x_i} . \quad (23)$$

Here we have used (22) and (16). We now set  $\underline{f} = (f_1, f_2, f_3)$  where

$$f_i = k_i / \omega = g_i / c^2 , \quad i = 1, 2, 3 . \quad (24)$$

Since  $\omega_{x_i} = -\phi_{tx_i} = -(k_i)_t$ ,  $\nabla\omega = -\underline{k}_t$ , hence

$$\underline{k}_t \cdot \underline{k} = \frac{1}{2} (k^2)_t = \frac{1}{2} (\omega^2/c^2)_t = c^{-2} \omega \omega_t. \quad (25)$$

Therefore

$$\begin{aligned} \nabla \cdot \underline{f} &= \frac{1}{\omega} \nabla \cdot \underline{k} - \frac{1}{\omega^2} \nabla\omega \cdot \underline{k} = \frac{1}{\omega} \nabla \cdot \underline{k} + \frac{1}{\omega^2} \underline{k}_t \cdot \underline{k} \\ &= \frac{1}{\omega} [\nabla \cdot \underline{k} + \frac{1}{2} \omega_t] = \frac{1}{\omega} [\phi_{x_v x_v} - \frac{1}{2} \phi_{tt}] = \frac{1}{\omega} L\phi. \end{aligned} \quad (26)$$

It follows from (23) that

$$\begin{aligned} \frac{d}{dt} \log |j| &= \frac{1}{j} \frac{dj}{dt} = \nabla \cdot \underline{g} = \nabla \cdot (c^2 \underline{f}) = c^2 \nabla \cdot \underline{f} + \underline{f} \cdot \nabla c^2 \\ &= \frac{c^2}{\omega} L\phi + \frac{1}{c^2} \underline{g} \cdot \nabla c^2 \end{aligned} \quad (27)$$

We now see from (16) that

$$\frac{c^2}{\omega} L\phi = \frac{d}{dt} \log |j| - \frac{1}{c^2} \frac{d}{dt} c^2 = \frac{d}{dt} \log \frac{|j|}{c^2} = \frac{d}{dt} \log y^2 = \frac{2}{y} \frac{dy}{dt}, \quad (28)$$

where

$$y = |j|^{\frac{1}{2}}/c. \quad (29)$$

By inserting (28) into (20) we obtain

$$\begin{aligned} \frac{d}{dt} [yz^{(m)}] &= y \left[ \frac{dz^{(m)}}{dt} + \frac{1}{y} \frac{dy}{dt} z^{(m)} \right] = y \left[ \frac{dz^{(m)}}{dt} + \left( \frac{c^2}{2\omega} L\phi \right) z^{(m)} \right] \\ &= - \frac{c^2 y}{2\omega} Lz^{(m-1)} \end{aligned} \quad (30)$$

This equation can now be integrated to yield  $z^{(m)}(t)$  in the form

$$y(t)z^{(m)}(t) = y(t_1)z^{(m)}(t_1) - \frac{1}{2\omega} \int_{t_1}^t c^2 y Lz^{(m-1)} dt' \quad (31)$$

Here we have not indicated explicitly the dependence on the parameters  $\underline{\sigma}$ . Thus, e.g.  $z^{(m)}(t) = z^{(m)}[t, \underline{x}(t, \underline{\sigma})]$  is the value at time  $t$  of  $z^{(m)}$  on the bicharacteristic  $[t, \underline{x}(t, \underline{\sigma})]$ .

If  $j(t)$  vanishes at  $t = t_0$ , the space-time point  $[t_0, \underline{x}(t_0, \underline{\sigma})]$  is called a caustic point. Then  $y(t_0)$  vanishes and  $z^{(m)}(t_0)$  becomes infinite. If  $t = t_1$  is a caustic point (31) is not valid because  $z^{(m)}(t_1)$  is infinite and, in general, the integral diverges. In order to obtain a valid formula for this case, we first define the "finite part" of a function. For  $t > t_0$  let  $f(t)$  have an asymptotic expansion in powers (perhaps fractional) of  $(t - t_0)$  as  $t \rightarrow t_0$ . Let  $f_\infty(t)$  denote the singular terms [negative powers of  $(t - t_0)$ ] in this expansion. We define the finite part of  $f(t)$  as  $t \rightarrow t_0$  by

$$\text{fin } f(t) = \lim_{t \rightarrow t_0} [f(t) - f_\infty(t)] \quad (32)$$



Now if  $\int_{t_0}^a g(x)dx$  is divergent or convergent at  $x = t_0$  we define the finite part of the integral as

$$\oint_{t_0}^a g(x)dx = \lim_{t \rightarrow t_0} \int_t^a g(x)dx . \quad (33)$$

We now set

$$\zeta^{(m)} = \lim_{t \rightarrow t_0} y(t)z^{(m)}(t) = \lim_{t \rightarrow t_0} \frac{|j(t)|^{\frac{1}{2}} z^{(m)}(t)}{c(t)} , \quad (34)$$

and take the finite part as  $t_1 \rightarrow t_0$  of (31). This yields

$$y(t)z^{(m)}(t) = \zeta^{(m)} - \frac{1}{2\omega} \oint_{t_0}^t c^2 y L z^{(m-1)} dt' , \quad (35)$$

or

$$z^{(m)}(t) = \frac{\zeta^{(m)} c(t)}{|j(t)|^{\frac{1}{2}}} - \frac{1}{2\omega} \oint_{t_0}^t c(t)c(t') \left| \frac{j(t')}{j(t)} \right|^{\frac{1}{2}} L z^{(m-1)}(t') dt' ,$$

$$m = 0, 1, 2, \dots; z^{(-1)} \equiv 0 . \quad (36)$$

If  $t = t_0$  is not a caustic point  $j(t)$  and  $z^{(m)}(t)$  are continuous there. Then (34) yields  $\zeta^{(m)} = y(t_0)z^{(m)}(t_0)$  and the finite part integral becomes an ordinary integral. In this case (35) or (36) reduces to (31) with  $t_1$  replaced by  $t_0$ .

In (36)  $c(t) = c[\underline{x}(t, \underline{\sigma})]$ . For the special case of a homogeneous medium  $c$  is constant and (36) becomes

$$z^{(m)}(t) = \frac{\delta^{(m)}}{|g(t)|^{\frac{1}{2}}} - \frac{c^2}{2\omega} \int_{t_0}^t \left| \frac{j(t')}{j(t)} \right|^{\frac{1}{2}} Lz^{(m-1)}(t') dt' ,$$

$$m = 0, 1, 2, \dots ; z^{(-1)} \equiv 0 . \quad (37)$$

Here

$$\delta^{(m)} = \lim_{t \rightarrow t_0} |j(t)|^{\frac{1}{2}} z^{(m)}(t) . \quad (38)$$

When  $c$  is constant we see from (16) that  $\underline{k}$  is constant on a bicharacteristic and from (17) that  $\underline{g}$  has the constant value

$$\underline{g} = c \underline{u}$$

where  $\underline{u}$  is the unit vector

$$\underline{u} = \frac{c}{\omega} \underline{k} . \quad (40)$$

Then the first equation (16) can be integrated to yield the explicit ray formula

$$\underline{x} = \underline{x}(t, \underline{\sigma}) = \underline{x}_0(\underline{\sigma}) + c[t - t_0(\underline{\sigma})]\underline{u}(\underline{\sigma}) . \quad (41)$$

The bicharacteristic  $[t, \underline{x}(t, \underline{\sigma})]$  is now a straight line in space time. It passes through the point  $[t_0(\underline{\sigma}), \underline{x}_0(\underline{\sigma})]$ .

### SECTION III

#### DIFFRACTION BY EDGES

Let

$$v_o(t, \underline{x}) = \sum_{m=0}^{\infty} e_m(\phi) z^{(m)} \quad (42)$$

be a progressing wave solution of (2) with constant  $c$ . Let  $S$  be a boundary surface in  $\underline{x}$ -space, such as the surface of a fixed scattering obstacle. In space-time the boundary is a hyper-cylinder  $\tilde{S}$  with generators parallel to the  $t$ -axis. The projection of  $\tilde{S}$  on  $\underline{x}$ -space is  $S$ . If the bicharacteristics of (42) intersect  $\tilde{S}$ , a reflected progressing wave is produced. Such problems are discussed in detail in [2]. In an important special case the incident wave is a plane wave

$$v_o(t, \underline{x}) = e_o(t - x_1/c) \quad (43)$$

where  $e_o$  is an arbitrary function. Then (43) is not only a special case of (42), but is in fact an exact solution of (2).

An edge  $E$  is a curve along which  $S$  fails to be smooth. Examples are the edge of a screen or the edge of a finite cylinder or other object which is locally wedge-shaped. If  $S$  contains an edge, we assume that in addition to the incident and reflected waves the solution contains a third term, the diffracted progressing wave

$$\hat{v}(t, \underline{x}) = \sum_{m=0}^{\infty} \hat{e}_m(\hat{\phi}) \hat{z}^{(m)} . \quad (44)$$

We assume that the phase function  $\hat{\phi}$  of the diffracted wave is equal to the phase function  $\phi$  of the incident wave on  $E$ . Let  $\eta$  be an arc-length parameter and  $\underline{x} = \underline{x}_0(\eta)$  the parametric equation of  $E$ . Then

$$\hat{\phi}[t, \underline{x}_0(\eta)] = \phi[t, \underline{x}_0(\eta)] \quad (45)$$

By differentiating (45) with respect to  $t$  and  $\eta$  we obtain

$$\hat{\omega} = \omega \quad \text{on } E , \quad (46)$$

and

$$\hat{\underline{k}} \cdot \underline{t}_1 = \underline{k} \cdot \underline{t}_1 \quad \text{on } E . \quad (47)$$

Here  $\hat{\omega} = -\hat{\phi}_t$ ,  $\hat{\underline{k}} = \nabla \hat{\phi}$ , and  $\underline{t}_1 = d\underline{x}_0/d\eta$  is the unit tangent vector to  $E$ . In terms of the unit vectors  $\underline{u} = \frac{c}{\omega} \underline{k}$  and  $\hat{\underline{u}} = \frac{c}{\hat{\omega}} \hat{\underline{k}}$ , which point in the directions of the incident and diffracted rays, (46) and (47) yield

$$\hat{\underline{u}} \cdot \underline{t}_1 = \cos \beta \quad (48)$$

where  $\beta = \beta(t, \eta)$  is the angle between the incident ray and the edge, i.e.

$$\cos \beta = \underline{u} \cdot \underline{t}_1 \quad (49)$$

This is the familiar law of edge diffraction (see [1]). If we introduce the unit normal and binormal vectors  $\underline{n}$  and  $\underline{b}$  of  $E$  we see from (48) that

$$\hat{\underline{u}} = \hat{\underline{u}}(t, \eta, \alpha) = \cos \beta \underline{t}_1 + \sin \beta \cos \alpha \underline{n} + \sin \beta \sin \alpha \underline{b} . \quad (50)$$

This equation defines the angle  $\alpha$ . In order to satisfy (45) we find that at each time  $t = \tau$  a one parameter family of diffracted bi-characteristics emanates from each point  $\underline{x}_0(\eta)$  of  $E$ . The corresponding rays are given by

$$\underline{x} = \underline{x}(t, \tau, \eta, \alpha) = \underline{x}_0(\eta) + c[t - \tau] \hat{\underline{u}}(\tau, \eta, \alpha), \quad 0 \leq \alpha < 2\pi . \quad (51)$$

They generate a cone with vertex at  $\underline{x}_0$  and semiangle  $\beta$ . Equation (51) is a special case of (41) with parameters

$$(\sigma_1, \sigma_2, \sigma_3) = (\tau, \eta, \alpha) . \quad (52)$$

From (18), (19), and (40) we find that on the bicharacteristic (51),

$$\hat{\omega} = \omega[\tau, \underline{x}_0(\eta)] \quad (53)$$

$$\hat{\phi} = \phi[\tau, \underline{x}_0(\eta)] \quad (54)$$

and

$$\hat{k} = \frac{\hat{\omega}}{c} \hat{u}(\tau, \eta, \alpha) . \quad (55)$$

In order to use (37) to determine the amplitude coefficients we must compute the jacobian

$$j(t) = j(t, \tau, \eta, \alpha) = \frac{\partial(x_1, x_2, x_3)}{\partial(\tau, \eta, \alpha)} \quad (56)$$

from (51). In so doing we make use of the Frenet equations,

$$t_1' = \kappa \underline{n} , \underline{n}' = -\kappa \underline{t}_1 + \tau_0 \underline{b} , \underline{b}' = -\tau_0 \underline{n} ; ' = \frac{d}{d\eta} . \quad (57)$$

Here  $\kappa$  and  $\tau_0$  are the curvature and torsion of the curve E. A straight-forward computation yields

$$j(t) = \omega \sin^2 \beta \left(1 + \frac{\sigma}{\rho}\right) = c^2 \sin^2 \beta (t - \tau) \left[1 + \frac{c(t - \tau)}{\rho}\right], \quad (58)$$

where

$$\sigma = c(t - \tau) \quad (59)$$

is the distance along a diffracted ray from the edge,

$$\rho = \frac{-\sin^2 \beta}{\beta' \sin \beta + \frac{\beta}{c} \sin \beta \cos \beta + \kappa \cos \beta} , \beta' = \beta_{\eta}(\tau, \eta), \dot{\beta} = \beta_{\tau}(\tau, \eta), \quad (60)$$

and

$$\cos\delta = \sin\beta\cos\alpha = \underline{\hat{u}} \cdot \underline{\hat{n}} . \quad (61)$$

From (37) with  $m = 0$ , and from (58) we now see that

$$\hat{z}^{(0)}(t) = \gamma^{(0)}(\tau, \eta, \alpha) \left| \sigma \left( 1 + \frac{\sigma}{\rho} \right) \right|^{-\frac{1}{2}}, \quad \sigma = c(t - \tau) . \quad (62)$$

Here we have set

$$\gamma^{(0)} = \frac{\delta^{(0)}}{c^{\frac{1}{2}} \sin\beta} . \quad (63)$$

Thus in order to obtain the leading term,

$$\hat{v}(t, \underline{x}) \sim \hat{e}_0(\phi) \hat{z}^{(0)} , \quad (64)$$

of the diffracted wave it remains to determine the functions  $\hat{e}_0(t)$  and  $\gamma^{(0)}(\tau, \eta, \alpha)$ . We assume that the amplitude  $\hat{z}^{(0)}$  of the diffracted wave is proportional to the amplitude  $z^{(0)}$  of the incident wave at the (space-time) point of diffraction. Thus we set

$$\gamma^{(0)}(\tau, \eta, \alpha) = D_0 z^{(0)}[\tau, \underline{x}_0(\eta)] . \quad (65)$$

Here the diffraction coefficient  $D_0$  is to be determined.

The value of  $D_0$  depends on the boundary condition at  $S$  and on the local geometry at the edge. We consider the two boundary conditions,

$$(A) \quad v = 0, \quad (B) \quad \frac{\partial v}{\partial n} = 0 \quad \text{on } S \quad (66)$$

Here  $\frac{\partial v}{\partial n}$  denotes the normal derivative. The local geometry is illustrated in Figure 1. The plane of the figure is orthogonal to the edge and the unit tangent vector  $\underline{t}_1$  points into the plane. The projections of the incident and diffracted rays onto the plane of the figure are shown.  $\gamma$  is the local wedge angle which is bounded by the tangents to  $S$  at  $E$ . The angles  $\theta$  and  $\alpha$  are simply related as shown in the figure, i.e.

$$\theta = \Psi - \alpha \quad (67)$$

where  $\Psi = \Psi(\eta)$  is the angle between the unit normal vector  $\underline{n}$  and the vector  $\underline{t}_3$  which is orthogonal to one tangent plane of  $S$  at  $\underline{x}_0(\eta)$ .

In order to compute  $\hat{e}_0$  and  $D_0$  we begin with an incident wave of the form given by (1) and (4). Then

$$v_0 \sim e^{i[ks(\underline{x}) - \omega t]} \sum_{m=0}^{\infty} (ik)^{-m} z_m(\underline{x}), \quad k = \omega/c, \quad (68)$$



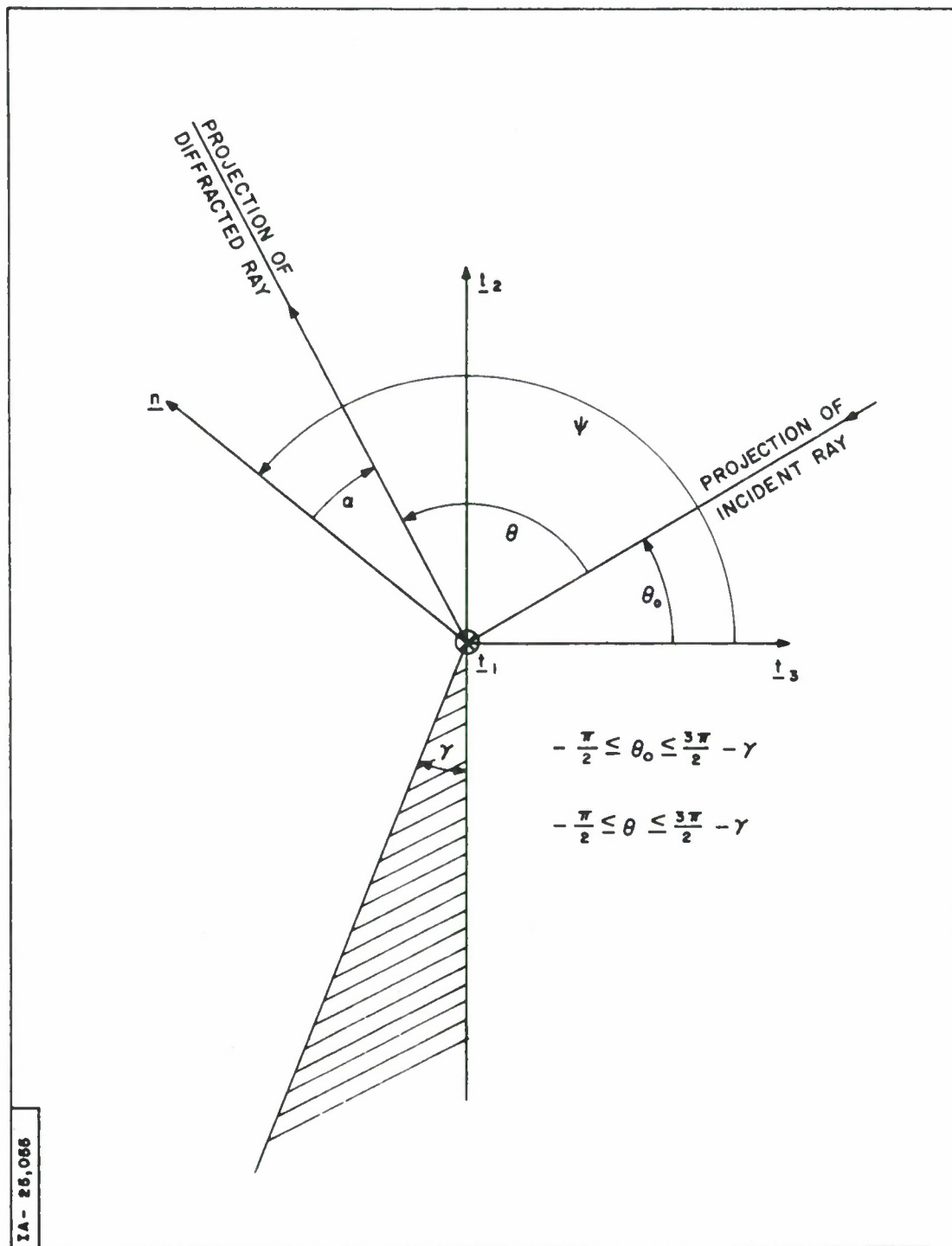


Figure 1. Rays of Edge Diffraction

and according to the geometrical theory of diffraction [1]

the diffracted wave is given by

$$\hat{v} \sim (k)^{-\frac{1}{2}} e^{i[k\hat{s}(\underline{x}) - \omega t]} \sum_{m=0}^{\infty} (ik)^{-m} \hat{z}_m(\underline{x}) . \quad (69)$$

The leading term is given by

$$\hat{z}_0 = e^{i\pi/4} D_{z_0} [x_0(\eta)] \left| \sigma \left( 1 + \frac{\sigma}{\rho} \right) \right|^{-\frac{1}{2}} , \quad (70)$$

where

$$D = \frac{\sin(\pi/q)}{(2\pi)^{\frac{1}{2}} q \sin \beta} \left[ \left( \cos \frac{\pi}{q} - \cos \frac{\theta - \theta_0}{q} \right)^{-1} \mp \left( \cos \frac{\pi}{q} - \cos \frac{\theta + \theta_0 + \pi}{q} \right)^{-1} \right] . \quad (71)$$

Here  $q = 2 - \frac{\gamma}{\pi}$  and  $\rho$  is given by (60) with  $\dot{\beta} = 0$ . The upper or lower sign in (71) holds for the boundary condition (66) (A) or (B).

We now set

$$\omega = b\nu , \quad \phi = b(t - s/c) , \quad \hat{\phi} = b(t - \hat{s}/c) \quad (72)$$

where  $b$  is a constant. Then  $k = b\nu/c$  and we may re-write (68) and (69) in the form

$$v_0 \sim e^{-i\nu\phi} \sum_{m=0}^{\infty} (-i\nu)^{-m} z^{(m)} , \quad z^{(m)} = \left(-\frac{c}{b}\right)^m \hat{z}_m(\underline{x}) ; \quad (73)$$

and

$$\hat{v} \sim (-iv)^{-\frac{1}{2}} e^{-i\sqrt{v}\phi} \sum_{m=0}^{\infty} (-iv)^{-m} \hat{z}^{(m)}, \quad \hat{z}^{(m)} = (-1)^m e^{-i\frac{\pi}{4}} \left(\frac{c}{b}\right)^{m+\frac{1}{2}} \hat{z}_m(\underline{x}). \quad (74)$$

Here  $(-iv)^{-\frac{1}{2}}$  denotes the principal value, i.e.  $(-iv)^{-\frac{1}{2}} = v^{-\frac{1}{2}} e^{i\pi/4}$ .

We now multiply the incident and diffracted waves (73) and (74) by an arbitrary function  $a(v)$  and integrate with respect to  $v$ . Then they become

$$v_o(t, \underline{x}) \sim \sum_{m=0}^{\infty} e_m(\phi) z^{(m)} \quad (75)$$

and

$$\hat{v}(t, \underline{x}) \sim \sum_{m=0}^{\infty} e_{m+\frac{1}{2}}(\phi) \hat{z}^{(m)} \quad (76)$$

where

$$e_r(t) = \int_0^{\infty} (-iv)^{-r} a(v) e^{-i\sqrt{v}t} dv. \quad (77)$$

We note that (77) holds for both integral and fractional values of  $r$ .

Thus (see the Appendix) for all  $r$

$$e_r(t) = I_r e_o(t) \quad (78)$$

where  $I_r$  is the fractional integral operator of order  $r$ .

By comparing (42) and (44) with (75) and (76) we conclude that in general

$$\hat{e}_m(t) = e_{m+\frac{1}{2}}(t) . \quad (79)$$

We also insert (65) into (62) and compare the result with the expression for  $\hat{z}^{(0)}$  obtained by inserting (70) into (74). Since  $z^{(0)} = z_0$  we conclude that

$$D_0 = \left(\frac{c}{b}\right)^{\frac{1}{2}} D = \left(\frac{c}{\phi_t}\right)^{\frac{1}{2}} D . \quad (80)$$

Summarizing our results we conclude that the leading term of the diffracted progressing wave is given by

$$\hat{v} \sim e_{\frac{1}{2}}(\hat{\phi}) \left(\frac{c}{\phi_t}\right)^{\frac{1}{2}} D z^{(0)}[\tau, \underline{x}_0(\eta)] \left|\sigma \left(1 + \frac{\sigma}{\rho}\right)\right|^{-\frac{1}{2}} , \quad (81)$$

where  $\sigma = c(t - \tau)$ ,  $\rho$  is given by (60) and  $D = D(\tau, \eta, \alpha)$  is given by (71) and (67). Thus  $\hat{v}(t, \underline{x})$  is given parametrically with parameters  $(\tau, \eta, \alpha)$  by (81) and (51).

## SECTION IV

### DIFFRACTION BY MOVING EDGES

The progressing wave formalism can be applied to problems of scattering by moving as well as fixed obstacles. The interesting problem of reflection of a progressing wave by a moving surface is discussed in [2]. There it was found that the usual law of reflection is modified, and a Doppler shift of the instantaneous frequency is introduced. It is also interesting to note that if a time-independent progressing wave such as (5) or (43) is scattered by a moving surface the reflected wave is, in general, not time-independent. In problems involving moving sources [2, 3, 4] general (i.e. not time-independent) progressing waves also arise naturally.

In this section we give a partial treatment of diffraction of a progressing wave by a moving edge such as the edge of a finite cylinder which is undergoing translation and rotation. As in the geometrical theory of diffraction, a full treatment would require the solution of a "canonical problem", involving a moving infinite wedge. However, considerable information can be obtained without solving that problem.

Let  $\eta$  be an arclength parameter on the edge E and let

$$\underline{x} = \underline{\xi}(\eta, t) \tag{82}$$

be the parametric equation of E at time t. Then  $\underline{t}_1 = \underline{\xi}_\eta$  is the instantaneous unit tangent vector to E. We introduce the vector

$$\underline{w} = \underline{\xi}_t - \underline{\xi}_t \cdot \underline{t}_1 \underline{t}_1 . \quad (83)$$

For any function  $\eta(t)$  the point

$$\underline{x} = \underline{x}(t) = \underline{\xi}[\eta(t), t] \quad (84)$$

is constrained to remain on the moving edge. Its velocity is

$\dot{\underline{x}} = \dot{\eta} \underline{t}_1 + \underline{\xi}_t$ , and the normal component of its velocity is

$$\dot{\underline{x}} - \dot{\underline{x}} \cdot \underline{t}_1 \underline{t}_1 = \dot{\eta} \underline{t}_1 + \underline{\xi}_t - \dot{\eta} \underline{t}_1 - \underline{\xi}_t \cdot \underline{t}_1 \underline{t}_1 = \underline{w}. \quad (85)$$

Therefore we refer to  $\underline{w}$  as the normal velocity vector of E and to its magnitude  $w = |\underline{w}|$  as the normal speed.

As in Section III we assume that the phase function  $\hat{\phi}$  of the diffracted wave (44) is equal to the phase function  $\phi$  of the incident wave (42) on E. Thus

$$\hat{\phi}[t, \underline{\xi}(\eta, t)] = \phi[t, \underline{\xi}(\eta, t)] , \quad (86)$$

and differentiation with respect to  $t$  and  $\eta$  yields

$$-\hat{\omega} + \hat{k} \cdot \underline{\xi}_t = -\omega + k \cdot \underline{\xi}_t \quad \text{on } E, \quad (87)$$

$$\hat{k} \cdot t_1 = k \cdot t_1 \quad \text{on } E. \quad (88)$$

If we multiply (88) by  $-(\underline{\xi}_t \cdot t_1)$  and add the result to (87) we obtain

$$-\hat{\omega} + \hat{k} \cdot \underline{w} = -\omega + k \cdot \underline{w} \quad \text{on } E. \quad (89)$$

In terms of the unit vectors  $\underline{u} = \frac{c}{\omega} \underline{k}$  and  $\underline{\underline{u}} = \frac{c}{\hat{\omega}} \hat{k}$ , (88) and (89) become

$$\hat{\omega} \underline{\underline{u}} \cdot t_1 = \omega \underline{u} \cdot t_1 \quad \text{on } E, \quad (90)$$

$$\hat{\omega} \left[ 1 - \frac{1}{c} \underline{\underline{u}} \cdot \underline{w} \right] = \omega \left[ 1 - \frac{1}{c} \underline{u} \cdot \underline{w} \right] \quad \text{on } E, \quad (91)$$

and it follows that

$$\frac{\hat{\omega} \underline{\underline{u}} \cdot t_1}{1 - \frac{1}{c} \underline{\underline{u}} \cdot \underline{w}} = \frac{\omega \underline{u} \cdot t_1}{1 - \frac{1}{c} \underline{u} \cdot \underline{w}} \quad \text{on } E. \quad (92)$$

We now introduce the unit vectors  $\underline{N} = \frac{1}{w} \underline{w}$ , and  $\underline{B} = \underline{t}_1 \times \underline{N}$ . Then  $\underline{t}_1$ ,  $\underline{N}$ ,  $\underline{B}$  form an orthonormal system but  $\underline{N}$  and  $\underline{B}$  are not the normal and binormal vectors in general. We also introduce the angles  $\beta$ ,  $\hat{\beta}$ ,  $\epsilon$ ,  $\hat{\epsilon}$  defined by

$$\underline{u} = \cos \beta \underline{t}_1 + \sin \beta \cos \epsilon \underline{N} + \sin \beta \sin \epsilon \underline{B}, \quad (93)$$

$$\hat{\underline{u}} = \cos \hat{\beta} \hat{\underline{t}}_1 + \sin \hat{\beta} \cos \hat{\epsilon} \hat{\underline{N}} + \sin \hat{\beta} \sin \hat{\epsilon} \hat{\underline{B}}. \quad (94)$$

Then (92) becomes

$$\frac{\cos \hat{\beta}}{1 - \frac{w}{c} \sin \hat{\beta} \cos \hat{\epsilon}} = \frac{\cos \beta}{1 - \frac{w}{c} \sin \beta \cos \epsilon}. \quad (95)$$

This equation may be called the law of diffraction for moving edges.

It reduces to the more familiar form (49) when  $w = 0$ . In general, for given  $\beta$ ,  $\epsilon$ , and  $w$  (95) determines  $\hat{\beta}$  as a function of  $\hat{\epsilon}$  (or vice-versa), hence a one-parameter family of directions  $\hat{\underline{u}}$  for the diffracted rays emanating from E. Furthermore (90) and (91) yield

$$\hat{\omega} = \mathcal{D}_w \quad (96)$$

where

$$\mathcal{D} = \frac{1 - \frac{1}{c} \underline{u} \cdot \underline{w}}{1 - \frac{1}{c} \hat{\underline{u}} \cdot \hat{\underline{w}}} = \frac{\cos \beta}{\cos \hat{\beta}} = \frac{1 - \frac{w}{c} \sin \beta \cos \epsilon}{1 - \frac{w}{c} \sin \hat{\beta} \cos \hat{\epsilon}}. \quad (97)$$



Since  $\omega$  and  $\hat{\omega}$  are proportional to the "instantaneous frequencies" of the incident and diffracted waves (see [2, 3, 4]), the factor  $\mathcal{D}$  may be viewed as determining a "Doppler shift". When  $w = 0$  we see that  $\mathcal{D} = 1$ .

It is interesting to note that back-scattering from the edge occurs when  $\hat{\underline{u}} = -\underline{u}$ . In this case we see from (90) and (91) that

$$(\omega + \hat{\omega}) \underline{u} \cdot \underline{t}_1 = 0 \quad \text{on } E, \quad (98)$$

and

$$\hat{\omega} = \omega \frac{1 - \frac{1}{c} \underline{u} \cdot \underline{w}}{1 + \frac{1}{c} \underline{u} \cdot \underline{w}} \quad \text{on } E. \quad (99)$$

Since  $w = |\underline{w}| < c$  we see from (99) that  $\omega$  and  $\hat{\omega}$  have the same sign, hence from (98) that

$$\underline{u} \cdot \underline{t}_1 = 0 \quad (100)$$

Thus, even for a moving edge, back-scattering occurs if and only if the edge is orthogonal to the incident ray.

## APPENDIX

### FRACTIONAL INTEGRALS AND FOURIER TRANSFORMS

For  $\sigma \geq 0$  we consider a class  $C_\sigma$  of functions  $f(t)$  which vanish for  $t$  less than some real number  $t_0$  and for which  $|f(t)| \leq c_0 e^{\sigma t}$  for sufficiently large positive  $t$ . Then the Fourier transform

$$a(\omega) = \frac{1}{2\pi} \int e^{i\omega t} f(t) dt \quad (101)$$

is analytic in  $I_m \omega > \sigma$ , and

$$f(t) = \int a(\omega) e^{-i\omega t} d\omega. \quad (102)$$

Here and hereafter the path of integration in the  $\omega$ -plane is parallel to the real axis but lies in the region  $I_m \omega > \sigma \geq 0$ . We introduce the linear operator

$$I_m f(t) = \int (-i\omega)^{-m} a(\omega) e^{-i\omega t} d\omega, \quad m \geq 0. \quad (103)$$

$I_m$  will be called the fractional integral of order  $m$ . If  $m$  is not an integer  $(-i\omega)^{-m}$  denotes the principal value.

#### Lemma 1

$$\text{For } m > 0, I_m f(t) = \frac{1}{\Gamma(m)} \int_{-\infty}^t (t - \tau)^{m-1} f(\tau) d\tau \quad (104)$$

where

$$\Gamma(m) = (m-1)! = \int_0^{\infty} e^{-z} z^{m-1} dz \quad (105)$$

Proof: Let

$$I = \int_{-\infty}^t (t - \tau)^{m-1} e^{-i\omega\tau} d\tau \quad (106)$$

and let  $\omega = re^{i\theta}$  where  $r = |\omega|$  and  $\theta = \arg \omega$ . We introduce the change of variable  $\tau = t + z/i\omega$  in (106). Then

$$I = \frac{e^{-i\omega t}}{(-i\omega)^m} \int_0^{\infty} e^{-z} z^{m-1} dz \quad (107)$$

For  $I_m$   $\omega > 0$ ,  $0 < \theta < \pi$  and  $-\pi/2 < \theta - \pi/2 < \pi/2$ . For values of  $\theta - \pi/2$  in this interval it is easy to show that the path of integration in (107) can be rotated to the positive real axis (i.e. the integral over the arc at infinity vanishes). Hence

$$I = (-i\omega)^{-m} e^{-i\omega t} \Gamma(m) \quad (108)$$

Thus

$$\begin{aligned} \frac{1}{\Gamma(m)} \int_{-\infty}^t (t - \tau)^{m-1} f(\tau) d\tau &= \frac{1}{\Gamma(m)} \int a(\omega) Id\omega \\ &= \int (-i\omega)^{-m} a(\omega) e^{-i\omega t} d\omega = I_m f(t) \end{aligned} \quad (109)$$

Lemma 2

$$I_n I_m = I_{n+m} . \quad (110)$$

proof: From (103), the Fourier transform of the function  $I_m f(t)$  is  $(-i\omega)^{-m} a(\omega)$ . Hence from (103)

$$I_n [I_m f(t)] = \int (-i\omega)^{-n} (-i\omega)^{-m} a(\omega) e^{-i\omega t} d\omega = I_{n+m} f(t) . \quad (111)$$

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13. ABSTRACT  For high frequency monochromatic waves the geometrical theory of diffraction can be used for predicting radar scattering. For typical targets the strongest returns, due to speculars, occur only at special aspect angles. At most aspect angles the dominant returns are usually due to diffraction by the "edges" of the target.  The proposed use of short pulse radars requires the consideration of scattering by non-monochromatic signals. The progressing wave formalism is a generalization of the geometrical theory of diffraction suitable for treating pulses for which the high frequencies predominate. This paper extends the earlier work on specular scattering of progressing waves to include scattering by edges. For a moving target, partial results are obtained.			

## KEY WORDS

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